# A MOCK FINAL EXAMINATION

1. This is an open-book examination. Only the official textbook (with no inserts) is allowed.

2. No notes or cheat-sheets of any kind allowed.

3. No calculators or computers of any kind allowed.

### Problem 1

Which properties of a natural cubic spline does the following posses, and which properties does it not possess?

$$f(x) = \begin{cases} (x+1) + (x+1)^3, & \text{if } x \in [-1, 0] \\ 4 + (x-1) + (x-1)^3, & \text{if } x \in (0, 1] \end{cases}$$

### Problem 2

Given a sufficiently smooth function $f : \mathcal{R} \to \mathcal{R}$,

1. use Taylor series to derive a fourth-order accurate formula for $f''(x)$ in terms of the values of $f(x)$, $f(x \pm h)$, and $f(x \pm 2h)$, with a chosen step size $h$.

2. Use the formula that you have just derived for $f''(x)$ to compute the first derivative of $\sin(x)$ at $x = 1$ using a step size of $h = 0.5$. Repeat the calculation using a step size of $h = 0.25$. Use Richardson extrapolation to produce a better estimate of the result. Do not actually compute any thing numerically. Just write down the necessary formulas and show how it can be done. Comment on the error that you get.

## Problem 3

In the above problem, if we use the derived formula to compute $f"(x)$ for a given step size $h$,

1. give an expression for the rounding error $E_{\text{round}}$ in terms of $h$ and the machine epsilon $\epsilon_{\text{machine}}$,
2. and an expression for the truncation error $E_{\text{trunc}}$ in terms of $h$.
3. Find the step size at which the total error is a minimum. Express it in terms of $\epsilon_{\text{machine}}$ and some constants of order unity.
4. Find this minimum error, and express it in terms of $\epsilon_{\text{machine}}$ and some constants of order unity.

For this problem, since we are not given the function $f(x)$, nor any knowledge of its derivatives, your expressions will have to written in terms of some unknown bounds and constants of order unity.

## Problem 4

Exercise 5.6 on p. 249 of Heath.

## Problem 5 (involving linear systems and linear least-squares)

Given a $3 \times 2$ matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0.5 & 1 \end{bmatrix}$$

and a $3 \times 1$ vector

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

we want to solve the linear least-squares problem

$$\mathbf{Ax} \cong \mathbf{b}$$

for the unknown vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1. Find the system of normal equations for this problem.

2. Solve this system of normal equations for $\mathbf{x}$ using Gaussian elimination. You have to specify the procedure that you follow to get the results. Do not use any other methods.

3. Next we want to solve the linear least-squares problem using the Householder $QR$ factorization. Compute $\mathbf{R}$ and the transformed $\mathbf{b}$ vector, but there is no need to compute $\mathbf{Q}$. Then use these results to compute $\mathbf{x}$. Does your answer here agree with the one obtained above using the normal equations?