SOLUTION FOR PROBLEM 5 OF THE MOCK EXAMINATION

Problem 5 (involving linear systems and linear least-squares)

Given a 3×2 matrix

$$\mathbf{A} = \begin{bmatrix} 0 & q \\ 0 & 0 \\ p & r \end{bmatrix}$$

and a 3×1 vector

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

we want to solve the linear least-squares problem

$$\mathbf{Ax} \cong \mathbf{b}$$

for the unknown vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1. The system of normal equations for this problem is

$$\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}.$$

We find

$$\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 0 & 0 & p \\ q & 0 & r \end{bmatrix} \begin{bmatrix} 0 & q \\ 0 & 0 \\ p & r \end{bmatrix} = \begin{bmatrix} p^2 & pr \\ pr & q^2 + r^2 \end{bmatrix}$$

and

$$\mathbf{A}^T\mathbf{b} = \begin{bmatrix} 0 & 0 & p \\ q & 0 & r \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} pb_3 \\ qb_1 + rb_3 \end{bmatrix}$$

The set of normal equations is therefore given by

$$\begin{bmatrix} p^2 & pr \\ pr & q^2 + r^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} pb_3 \\ qb_1 + rb_3 \end{bmatrix}$$

2. Next we solve the above normal equations using Gaussian elimination. Thus we want to replace the second row by

$$(\text{row } 2) - \frac{pr}{p^2} \times (\text{row } 1) \rightarrow \begin{bmatrix} 0 & q^2 \end{bmatrix}.$$

The second element of vector $\mathbf{b}$ must also be transformed exactly the same way:

$$(\text{element } 2) - \frac{pr}{p^2} \times (\text{element } 1) \rightarrow qb_1 + rb_3 - rb_3 = qb_1.$$

We have now transformed the system to upper triangular form

$$\begin{bmatrix} p^2 & pr \\ 0 & q^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} pb_3 \\ qb_1 \end{bmatrix}.$$

This can be solved by back-substitution:

$$q^2 x_2 = qb_1$$

and so $x_2 = \frac{b_1}{q}$. Substituting this into the first equation gives

$$p^2 x_1 + pr\frac{b_1}{q} = pb_3,$$

from which we have $x_1 = \frac{b_3}{p} - \frac{rb_1}{pq}$. Thus the solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{b_3}{p} - \frac{rb_1}{pq} \\ \frac{b_1}{q} \end{bmatrix}.$$

3. Next we want to solve the linear least-squares problem using the Householder QR factorization. First let us work on the first column of $\mathbf{A}$, which we denote by $\mathbf{u}$

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}.$$

The Euclidean norm of that vector is clearly given by p. Since the first element of $\mathbf{u}$ is zero, to obtain α in Heath's book, the overall sign for α really does not matter (no cancellation can occur either way). So we choose $\alpha = p$. The vector in Householder's matrix is then given by

$$\mathbf{v}_1 = \mathbf{u} - \alpha\mathbf{e}_1 = \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix} - \begin{bmatrix} p \\ 0 \\ 0 \end{bmatrix} = p\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Recall that the overall scale for vector $\mathbf{v}$ is irrelevant, so we can drop p and let

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Recall that if $\mathbf{v}$ is the vector defining the Householder matrix, and if $\mathbf{u}$ is a column of $\mathbf{A}$ that we want this Householder transformation to operate on, then

$$\mathbf{H}\mathbf{u} = \mathbf{u} - 2\frac{\mathbf{v}^T\mathbf{u}}{\mathbf{v}^T\mathbf{v}}\mathbf{v}.$$

Note that $||v_1||_2^2 = 2$. By design, using this $\mathbf{v}_1$ in defining the Householder matrix transforms the first column of $\mathbf{A}$ to $\begin{bmatrix} p \\ 0 \\ 0 \end{bmatrix}$, so we can simply skip the algebra. However we need to see how it transforms the second column of $\mathbf{A}$. So now we let

$$\mathbf{u} = \begin{bmatrix} q \\ 0 \\ r \end{bmatrix}.$$

We find

$$\mathbf{v}^T\mathbf{u} = r - q.$$

Thus

$$\mathbf{H}_1\mathbf{u} = \begin{bmatrix} q \\ 0 \\ r \end{bmatrix} - \frac{2(r-q)}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} r \\ 0 \\ q \end{bmatrix}.$$

Thus

$$\mathbf{H}_1\mathbf{A} = \begin{bmatrix} p & r \\ 0 & 0 \\ 0 & q \end{bmatrix}.$$

We also need to transform vector $\mathbf{b}$. We find

$$\mathbf{v}^T\mathbf{b} = b_3 - b_1.$$

Thus

$$\mathbf{H}_1\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} - \frac{2(b_3-b_1)}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_3 \\ 0 \\ b_1 \end{bmatrix}.$$

We now need to repeat the above procedure to eliminant the elements below the main diagonal of the second column of $\mathbf{H}_1\mathbf{A}$. To find α we must ignore the first element of that column. The result is $\alpha = q$.

The vector in Householder's matrix is then given by

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ q \end{bmatrix} - \begin{bmatrix} 0 \\ q \\ 0 \end{bmatrix} = q \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Again we can drop the scale factor q to write

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

Note that $||v_2||_2^2 = 2$.

So letting

$$\mathbf{u} = \begin{bmatrix} r \\ 0 \\ q \end{bmatrix}.$$

We find that

$$\mathbf{v}_2{}^T\mathbf{u} = q,$$

and so

$$\mathbf{H}_2\mathbf{u} = \begin{bmatrix} r \\ 0 \\ q \end{bmatrix} - \frac{2q}{2} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} r \\ q \\ 0 \end{bmatrix}.$$

Therefore we have

$$\mathbf{H}_2\mathbf{H}_1\mathbf{A} = \begin{bmatrix} p & r \\ 0 & q \\ 0 & 0 \end{bmatrix}.$$

From this equation we identify

$$\mathbf{R} = \begin{bmatrix} p & r \\ 0 & q \end{bmatrix}.$$

We also need to transform the vector $\mathbf{H}_1\mathbf{b}$. We find

$$\mathbf{v}^T\mathbf{H}_1\mathbf{b} = b_1.$$

Thus

$$\mathbf{H}_2\mathbf{H}_1\mathbf{b} = \begin{bmatrix} b_3 \\ 0 \\ b_1 \end{bmatrix} - \frac{2b_1}{2}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} b_3 \\ b_1 \\ 0 \end{bmatrix}.$$

Therefore the solution of the linear least-squares problem obeys the equation

$$\begin{bmatrix} p & r \\ 0 & q \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_3 \\ b_1 \end{bmatrix}.$$

This can be solved by back-substitution:

$$qx_2 = b_1$$

and so $x_2 = \frac{b_1}{q}$. Substituting this into the first equation gives

$$px_1 + r\frac{b_1}{q} = b_3,$$

from which we have $x_1 = \frac{b_3}{p} - \frac{rb_1}{pq}$. Thus the solution is

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{b_3}{p} - \frac{rb_1}{pq} \\ \frac{b_1}{q} \end{bmatrix}.$$

We see that the solution is exactly the same as what we obtained before by solving the normal equations.

We can also calculate the residual vector, since

$$\mathbf{Ax} = \begin{bmatrix} 0 & q \\ 0 & 0 \\ p & r \end{bmatrix}\begin{bmatrix} \frac{b_3}{p} - \frac{rb_1}{pq} \\ \frac{b_1}{q} \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \\ b_3 \end{bmatrix}$$

Therefore

$$\mathbf{r} = \mathbf{Ax} - \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}\begin{bmatrix} b_1 \\ 0 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 \\ b_2 \\ 0 \end{bmatrix},$$

thus $||r||_2 = |b_2|$.