

ASSIGNMENT 6 - Solution

Due March 25, 2004 (before start of class)

Problem 6: Computer Problem 5.2 on p.250 of Heath - Solution

Convergence properties of each iteration scheme for the root $x = 2$ are determined by comparing $|g'_i(2)|$ with unity. For $g_1(x)$, we have $g'_1(x) = 2x/3$, and so $g'_1(2) = 4/3$. Since $|g'_1(2)| > 1$, fix-point iteration using $g_1(x)$ will diverge.

For $g_2(x)$, we have $g'_2(x) = 3/(2\sqrt{3x-2})$, and so $g'_2(2) = 3/4$. Since $|g'_2(2)| < 1$, fix-point iteration using $g_2(x)$ is expected to converge to the correct root if the starting guess is not too far way.

For $g_3(x)$, we have $g'_3(x) = 2/x^2$, and so $g'_3(2) = 1/2$. Since $|g'_3(2)| < 1$, fix-point iteration using $g_2(x)$ is expected to converge to the correct root if the starting guess is not too far way.

For $g_4(x)$, we find that $g'_4(x) = 2(x-1)(x-2)/(2x-3)$, and so $g'_4(2) = 0$. Since $|g'_4(2)| = 0$, fix-point iteration using $g_2(x)$ is expected to converge at least quadratically to the correct root if the starting guess is not too far way.

Next we implement each of the schemes using the initial guess of $x = 4$.

The result for 12 iterations using $g_1(x)$ is:

iteration	x	error
1	6.000000000000000e+000	4.000000000000000e+000
2	1.266666666666667e+001	1.066666666666667e+001
3	5.414814814814814e+001	5.214814814814814e+001
4	9.78007315957933e+002	9.76007315957933e+002
5	3.18833436689080e+005	3.18831436689080e+005
6	3.38849201176566e+010	3.38849201156566e+010
7	3.82729270459989e+020	3.82729270459989e+020
8	4.88272314889451e+040	4.88272314889451e+040
9	7.94699511625012e+080	7.94699511625012e+080
10	2.10515771259011e+161	2.10515771259011e+161
11	Inf	Inf
12	Inf	Inf

The error keeps on increasing as x moves farther and farther to the right from the root at $x = 2$. If we had started with an initial guess of less than 2, then x will move to the left of the root at $x = 2$ but will converge to the other root at $x = 1$.

Notice that $g_1'(1) = 2/3$ and so we have convergence to the root at $x = 1$ since $|g_1'(1)| < 1$.

The result for 27 iterations using $g_2(x)$ is:

iteration	x	error
1	3.16227766016838e+000	1.16227766016838e+000
2	2.73620777363583e+000	7.36207773635829e-001
3	2.49171092242007e+000	4.91710922420072e-001
4	2.33990016181465e+000	3.39900161814648e-001
5	2.24046880929951e+000	2.40468809299505e-001
6	2.17287975458803e+000	1.72879754588025e-001
7	2.12570912021473e+000	1.25709120214728e-001
8	2.09215854099162e+000	9.21585409916204e-002
9	2.06796412516631e+000	6.79641251663097e-002
10	2.05033957565544e+000	5.03395756554399e-002
11	2.03740490010364e+000	3.74049001036392e-002
12	2.02785963525854e+000	2.78596352585447e-002
13	2.02078670467114e+000	2.07867046711372e-002
14	2.01552973533347e+000	1.55297353334709e-002
15	2.01161358267447e+000	1.16135826744688e-002
16	2.00869130232184e+000	8.69130232183890e-003
17	2.00650788858791e+000	6.50788858791085e-003
18	2.00487497509539e+000	4.87497509538803e-003
19	2.00365289541032e+000	3.65289541032121e-003
20	2.00273779767372e+000	2.73779767371529e-003
21	2.00205229527631e+000	2.05229527631134e-003
22	2.00153862961196e+000	1.53862961196305e-003
23	2.00115363948796e+000	1.15363948795544e-003
24	2.00086504254132e+000	8.65042541316807e-004
25	2.00064867671062e+000	6.48676710618634e-004
26	2.00048644837496e+000	4.86448374958925e-004
27	2.00036480301091e+000	3.64803010910020e-004

The convergence is very slow because $g_2'(2) = 3/4$. The error at each iteration is about $3/4$ of the error in the previous step.

The result for 15 iterations using $g_3(x)$ is:

iteration	x	error
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1	2.500000000000000e+000	5.000000000000000e-001
2	2.200000000000000e+000	2.000000000000000e-001
3	2.090909090909090e+000	9.09090909090908e-002
4	2.04347826086957e+000	4.34782608695654e-002
5	2.02127659574468e+000	2.12765957446810e-002
6	2.01052631578947e+000	1.05263157894737e-002
7	2.00523560209424e+000	5.23560209424101e-003
8	2.00261096605744e+000	2.61096605744138e-003
9	2.00130378096480e+000	1.30378096479777e-003
10	2.00065146579805e+000	6.51465798045425e-004
11	2.00032562683165e+000	3.25626831650716e-004
12	2.00016278691193e+000	1.62786911932056e-004
13	2.00008138683161e+000	8.13868316105193e-005
14	2.00004069175992e+000	4.06917599184986e-005
15	2.00002034546601e+000	2.03454660128166e-005

The convergence is faster because $g_2'(2) = 1/2$. The error at each iteration is about 1/2 of the error in the previous step.

The result for 7 iterations using $g_4(x)$ is:

iteration	x	error
1	2.800000000000000e+000	8.000000000000000e-001
2	2.24615384615385e+000	2.46153846153846e-001
3	2.04060269627280e+000	4.06026962727992e-002
4	2.00152476019449e+000	1.52476019448988e-003
5	2.00000231782540e+000	2.31782539517056e-006
6	2.00000000000537e+000	5.37259126076606e-012
7	2.00000000000000e+000	0.00000000000000e+000

The convergence is very fast because of the quadratic convergence.