ASSIGNMENT 10 - Solution

Problem 10: - Solution

We want to derive a 4th-order formula for the first derivative of a function at x in terms of the values of $f(x)$, $f(x \pm h)$, and $f(x \pm 2h)$ for a given step size h. We will make use of the following Taylor expansion:

$$f(x \pm h) = f(x) \pm f'(x)h + \frac{1}{2}f''(x)h^2 \pm \frac{1}{6}f^{(3)}(x)h^3 + \frac{1}{24}f^{(4)}(x)h^4 + \mathcal{O}h^5$$

$$f(x \pm 2h) = f(x) \pm 2f'(x)h + 2f''(x)h^2 \pm \frac{4}{3}f^{(3)}(x)h^3 + \frac{2}{3}f^{(4)}(x)h^4 + \mathcal{O}h^5$$

assuming that h is sufficiently small.

First notice that in order to derive a formula for $f'(x)$, we will need to divide by h eventually. Therefore terms of $\mathcal{O}h^5$ or higher can be dropped from the Taylor expansions. However all terms lower than $\mathcal{O}h^5$ must be kept. Clearly we do not want to have to compute $f''(x)$, $f^{(3)}(x)$, or $f^{(4)}(x)$. Our strategy is to eliminate terms involving these derivatives among the above 4 equations.

Terms involving even derivatives of $f(x)$ can easily be eliminated since

$$f(x+h)f(x-h) = 2f'(x)h + \frac{1}{3}f^{(3)}(x)h^3 + \mathcal{O}h^5$$

$$f(x+2h)f(x-2h) = 4f'(x)h + \frac{8}{3}f^{(3)}(x)h^3 + \mathcal{O}h^5$$

Now we need to eliminate the $f^{(3)}(x)$ term. This can be done by taking 8 times the first equation and subtracting off the second equation. Solving for $f'(x)$ from the resulting expressing gives us the following 4th order finite difference formula for the first derivative:

$$f'(x) = \frac{1}{12h}\left[-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)\right] + \mathcal{O}h^4.$$

We now use the above formula to compute the first derivative of the sine function at $x = 1$ For $h = 0.5$, we obtain $f'(1) = 5.392096928621882e - 001$ (an error of $-1.092613005951537e - 003$. If we reduce the step size by a half, we find $f'(1) = 5.402324755527217e - 001$ (an error of $-6.983031541807350e - 005$.

The error for the Richardson extrapolation of course depends on the number of levels of iteration (controlled by `itMax` in the MATLAB program). One can achieve an error of as small as 10^{-14} by using an `itMax` of 10.