A few Methods for Computing π

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Abstract: We briefly mention a few fast-converging algorithms available in the literature for computing the value of π.

Directory

Methods for computing the value of π is probably the most celebrated evaluation problem in the history of mankind. The problem dates back as early as the Babylonians and then to the Greek mathematician Archimedes (around 250 BC) who was able to provide insights and at least three significant digits for π.

In order to be able to compute π to a huge number of digits, one must be able to perform extremely high precision arithmetic, and to compute these digits using fast algorithms or special series.

There are two interesting series by the Indian mathematician Srinivasa Ramanujan (1887-1920) around 1910. The first one

$$\frac{4}{\pi} = \sum_{n=0}^{\infty} \frac{(-1)^n (1123 + 21460n)(2n-1)!!(4n-1)!!}{882^{2n+1} \, 32^n (n!)^3}$$

gives 6 decimal places per term, and the second one

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} (4n)! \frac{1103 + 26390n}{(n!)^4 \, 396^{4n}}$$

gives 8 decimal places per term.

In 1994 David and Gregory Chudnovsky computed π to billion decimal places using the series

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)! (13591409 + 545140134n)}{(3n)! \, (n!)^3 \, 640320^{3n+3/2}}.$$

This series yields 14 decimal digits per term.

For all these series, the number of terms one must compute increases linearly with the number of digits desired in the result. In other words, if one wishes to compute π to twice as many digits, then one must evaluate twice as many terms of the series.

In 1976 Eugene Salamin and Richard Brent independently discover a new formula based on the arithmetic-geometric mean to compute π. This scheme produces approximations that converge to π much faster than any classical method. The algorithm can be stated as follows. Set $a_0 = 1, b_0 = 1/\sqrt{2}$ and $s_0 = 1/2$. For $k = 1, 2, \ldots$ compute

$$a_k = \frac{a_{k-1} + b_{k-1}}{2}, \qquad b_k = \sqrt{a_{k-1} b_{k-1}},$$

$$s_k = s_{k-1} - 2^k(a_k^2 - b_k^2), \qquad p_k = \frac{2a_k^2}{s_k}.$$

Then p_k converges quadratically to π. This means that each iteration of this algorithm approximately doubles the number of correct digits. To be specific, successive iterations produce 1, 4, 9, 20, 42, 85, 173, 347 and 697 correct digits of π. Twenty five iterations are sufficient to compute to over 45 million decimal digit accuracy. However each of these iterations must be performed using a level of numeric precision that is at least as high as that desired for the final result.

Arithmetic-geometric mean schemes that give cubic (the number of good digits triples with every iteration) as well as quartic (the number of good digits quadruples with every iteration) convergence for the value of π have also been discovered. The quartic scheme yields 6 billion digits of π, and produces the current record holder.