POLYTECHNIC UNIVERSITY
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Applications Solving Linear Systems

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Abstract: The Hill cipher in cryptography is used to illustrate the application of matrices defined over a finite field, and the handling of characters and strings in computer programs.

Directory

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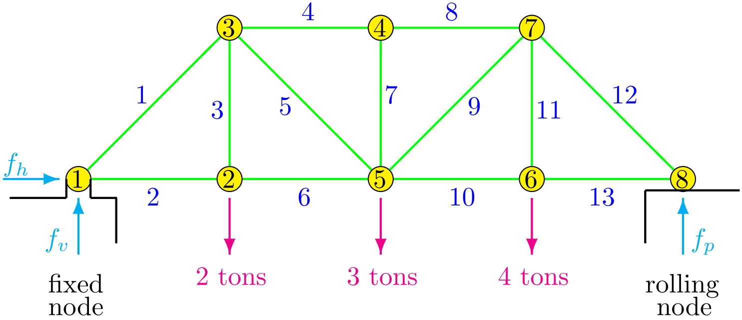
1. Statics of A Planar Truss

Trusses are composed of straight members of negligible masses connected at their ends by hinged connections to form a stable configuration. When loads are applied to a truss only at the joints, forces are transmitted only in the direction of each of its members. That is, the members experience tension or compression forces, but not bending forces. Trusses have a high strength to weight ratio and consequently are used in many structures, from bridges, to roof supports, to space stations.

The figure depicts a planar truss having 13 members (the numbered lines representing beams) connecting 8 joints (the numbered circles). The indicated loads, in tons, are applied at joints 2, 5, and 6. The beams are assumed to have negligible mass.

The magnitude of the force exerted by beam n is denoted by f_n, where n goes from 1 to 13. We assume that each beam experiences a tensile force, that is each beam is being stretched by the joints at each end. Under this assumption, in a free-body diagram of a given joint, each member connected to the joint must therefore apply a force directed *away* from the joint. If later our solution indicates that a particular force has a negative value then that means that the force is actually compressional meaning that the beam is being compressed by the 2 joints at each end. Computing the magnitude of these stresses and whether they are compression or tension is essential to designing the truss.



For the truss to be in static equilibrium, there must be no net force, horizontally or vertically, at any joint. Notice that each beam that is connected to a joint exerts no torque about that joint since the force due to the beam goes through the joint (the lever-arm is zero). Thus, we can determine the member forces by equating the horizontal forces to the left and right at each joint, and similarly equating the vertical forces upward and downward at each joint. For the eight joints, this would give 16 equations, which is more than the 13 unknown factors to be determined. For the truss to be statically determinate, that is, for there to be a unique solution, we assume that joint 1 is rigidly fixed both horizontally and vertically, and that joint 8 is fixed vertically. That means that joint 1 is held fixed to an anchor, and joint 8 is resting on a rigid platform. Thus there could be a horizontal and a vertical force, f_h and f_v, exerted by the anchor on joint 1 as shown in the figure. There must also be a vertical force f_p exerted on joint 8.

The member forces are resolved into horizontal and vertical components. A horizontal component of the force pointing to the right is taken to be positive. The same is true for a vertical component of the force that is pointing up. Defining $\alpha = 1/\sqrt{2}$, and using the fact that both the vertical and horizontal components of the force must add to zero at each joint, we obtain the following system of equations for the

16 member forces, $f_v, f_v, f_p, f_1, f_2, \ldots, f_{13}$. We have 2 equations for each joint, one for the horizontal forces and the other for the vertical forces, thus a total of 16 equations.

$$
\begin{array}{lcc}
\text{Joint 1:} & -f_h + f_2 = 0 & f_v + \alpha f_1 = 0 \\
\text{Joint 2:} & f_2 - f_6 = 0 & f_3 - 2 = 0 \\
\text{Joint 3:} & -\alpha f_1 + f_4 + \alpha f_5 = 0 & -\alpha f_1 - f_3 - \alpha f_5 = 0 \\
\text{Joint 4:} & -f_4 + f_8 = 0 & -f_7 = 0 \\
\text{Joint 5:} & -\alpha f_5 - f_6 + \alpha f_9 + f_{10} = 0 & \alpha f_5 + f_7 + \alpha f_9 - 3 = 0 \\
\text{Joint 6:} & -f_{10} + f_{13} = 0 & f_{11} - 4 = 0 \\
\text{Joint 7:} & -f_8 - \alpha f_9 + \alpha f_{12} = 0 & -\alpha f_9 - f_{11} - \alpha f_{12} = 0 \\
\text{Joint 8:} & -f_{13} - \alpha f_{12} = 0 & f_p + \alpha f_{12} = 0.
\end{array}
$$

With 16 linear equations involving 16 unknowns member forces, we have a system of linear equations (involving a 16 by 16 matrix) which can be solved to find all the member forces.

If we are not interested in the forces f_h, f_v and f_p, then we can omit the 2 equations for joint 1, and the second equation for joint 8. We then have a linear system of equations involving a 13 by 13 matrix for $f_1, f_2, \ldots, f_{13}$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 0 & -1 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \alpha & 0 & 1 & 0 & \alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 1 & 0 & 0 & -\alpha & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & 0 & 1 & 0 & \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha & 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 0 & 1 & \alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha & 1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \\ f_9 \\ f_{10} \\ f_{11} \\ f_{12} \\ f_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \\ 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

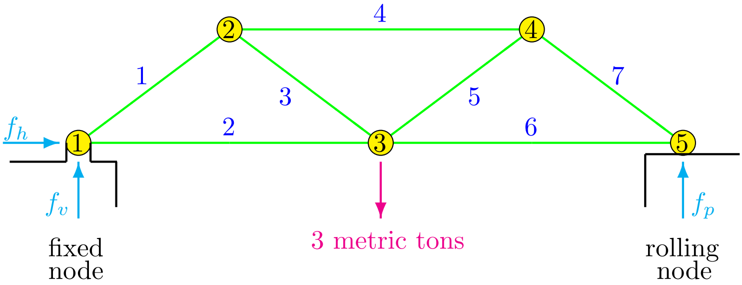
We can use MATLAB's backlash operator to solve the above linear system to obtain the vector $\mathbf{f} = [f_1 f_2 \ldots f_{13}]^T$, where T denotes the transpose operation. The computed results should agree very well with the exact solution

$$\mathbf{f} = [-4/\alpha,\ 4,\ 2,\ -6,\ 2/\alpha,\ 4,\ 0,\ -6,\ 1/\alpha,\ 5,\ 4,\ -5/\alpha,\ 5]^T .$$

One can easily show that under the present assumptions, f_7 is always identically zero no matter what the magnitudes of the forces applied at joints 2, 5 and 6 are. That means that we can remove beam 7 from the structure without altering its structural characteristics. Moreover there can be no horizontal force exerted on joint 1, *i.e.* f_h is always equal to zero.

The figure depicts a planar truss having 7 members (the numbered lines representing beams) connecting 5 joints (the numbered circles). A load of 3 metric tons is applied at joint 3. The beams are assumed to have negligible mass. Node 1 is held fixed while node 5 is supported on a rigid platform.

The length of a horizontal beam is $8m$, and the length of a slanted beam is $5m$. For each of the beams, compute the magnitude of the stress, and indicate whether the stress is compressional or tensile.


4
2
4
1
3
5
7
2
6
f_h
1
3
5
f_v
f_p
fixed
node
3 metric tons
rolling
node

References

[1] Most: the materials here are adopted from C. Moler, *Numerical Computing with Matlab* at the *Mathworks site*.

[2] G. Birkhoff, G. and S. MacLane, *A Survey of Modern Algebra*, 5th edition New York: Macmillan, p. 413, 1996.

[3] *D. Stinson Cryptographt: Theory and Practice*, 2nd edition, CRC press, 2002.