Forward Difference Formula for the First Derivative

We want to derive a formula that can be used to compute the first derivative of a function at any given point. Our interest here is to obtain the so-called forward difference formula. We start with the Taylor expansion of the function about the point of interest, x,

$$f(x+h) \approx f(x) + f'(x)h + \frac{f''(x)h^2}{2} + \ldots,$$

assuming that h is small. Solving for $f'(x)$ gives the formula for the forward difference scheme:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} - \frac{f''(x)h}{2} + \ldots .$$

The forward difference formula is a first order scheme since the error goes as the first power of h. The truncation error is bounded by $Mh/2$ where M is a bound on $|f''(t)|$ for t near x. Thus the formula is more and more accurate with decreasing h since the truncation error is then smaller.

However one must also consider the effect of rounding error. Assuming that rounding errors in computing the function values are bounded by the machine ϵ, then the rounding error in evaluating the above formula is $2\epsilon/h$. Thus rounding error increases with decreasing h.

The total computational error, E, is therefore bounded by the sum of these two errors

$$E = \frac{Mh}{2} + \frac{2\epsilon}{h}.$$

Since the first term coming from truncation decreases with decreasing h and the second term coming from rounding increases with decreasing h, there must be an optimal value for h that represents the best tradeoffs between these two sources of error and gives the smallest total error. To find this optimal value we differentiate E and set it to zero:

$$\frac{dE}{dh} = \frac{M}{2} - \frac{2\epsilon}{h^2} = 0.$$

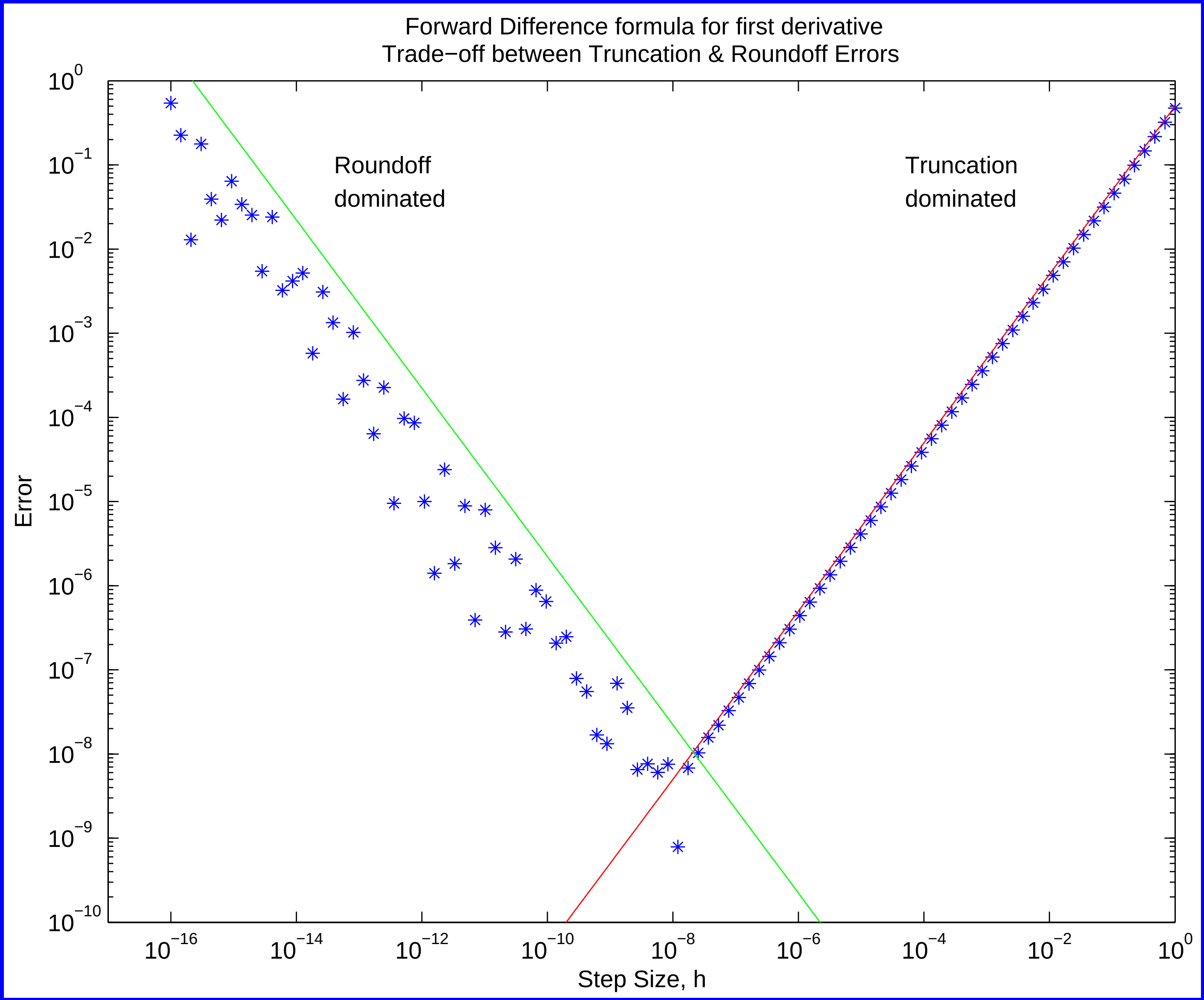
Solving for h gives the optimal value

$$h_{\text{min}} = 2\sqrt{\frac{\epsilon}{M}}.$$

Inserting this optimal value for h into the expression for E gives the minimum error that is achieved using this optimal h:

$$\begin{aligned} E_{\text{min}} &= \frac{M}{2} 2\sqrt{\frac{\epsilon}{M}} + 2\epsilon\frac{1}{2}\sqrt{\frac{M}{\epsilon}} \qquad (1) \\ &= \sqrt{M\epsilon} + \sqrt{M\epsilon} = 2\sqrt{M\epsilon}. \end{aligned}$$

Notice that truncation and rounding errors contribute equally to this total minimum computational error.


Forward Difference formula for first derivative
Trade−off between Truncation & Roundoff Errors
Roundoff
dominated
Truncation
dominated
Error
Step Size, h
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
10^{-8}
10^{-9}
10^{-10}
10^{-16}
10^{-14}
10^{-12}
10^{-10}
10^{-8}
10^{-6}
10^{-4}
10^{-2}
10^{0}