Computer Problem 6.5

A projectile is thrown with an initial speed of $\mathbf{v}$ at an angle α measured from the horizontal direction at a target located at a height h a distance x away. The horizontal component of the velocity is $v\cos(\alpha)$ and the vertical component is $v\sin(\alpha)$. Let t denotes the total time the projectile is in air.

We recall that the equation for the displacement Δx of an object under constant acceleration is

$$\Delta x = v_0 t + \frac{1}{2}at^2,$$

where v_0 is the initial speed, and a is the acceleration of the object. Using this equation for the horizontal motion of the projectile, we have

$$x = v\cos(\alpha)t,$$

because there is no acceleration horizontally. For the vertical motion, we have

$$h = v\sin(\alpha)t - \frac{1}{2}gt^2,$$

since the vertical acceleration is $-g$. Solving for t and then substituting it into the second equation gives

$$\frac{g}{2v^2\cos^2(\alpha)}x^2 - \tan(\alpha)x + h = 0.$$

Solving this quadratic equation gives

$$x = \frac{\tan(\alpha) \pm \sqrt{\tan^2(\alpha) - \frac{2gh}{v^2\cos^2(\alpha)}}}{\frac{g}{v^2\cos^2(\alpha)}}$$

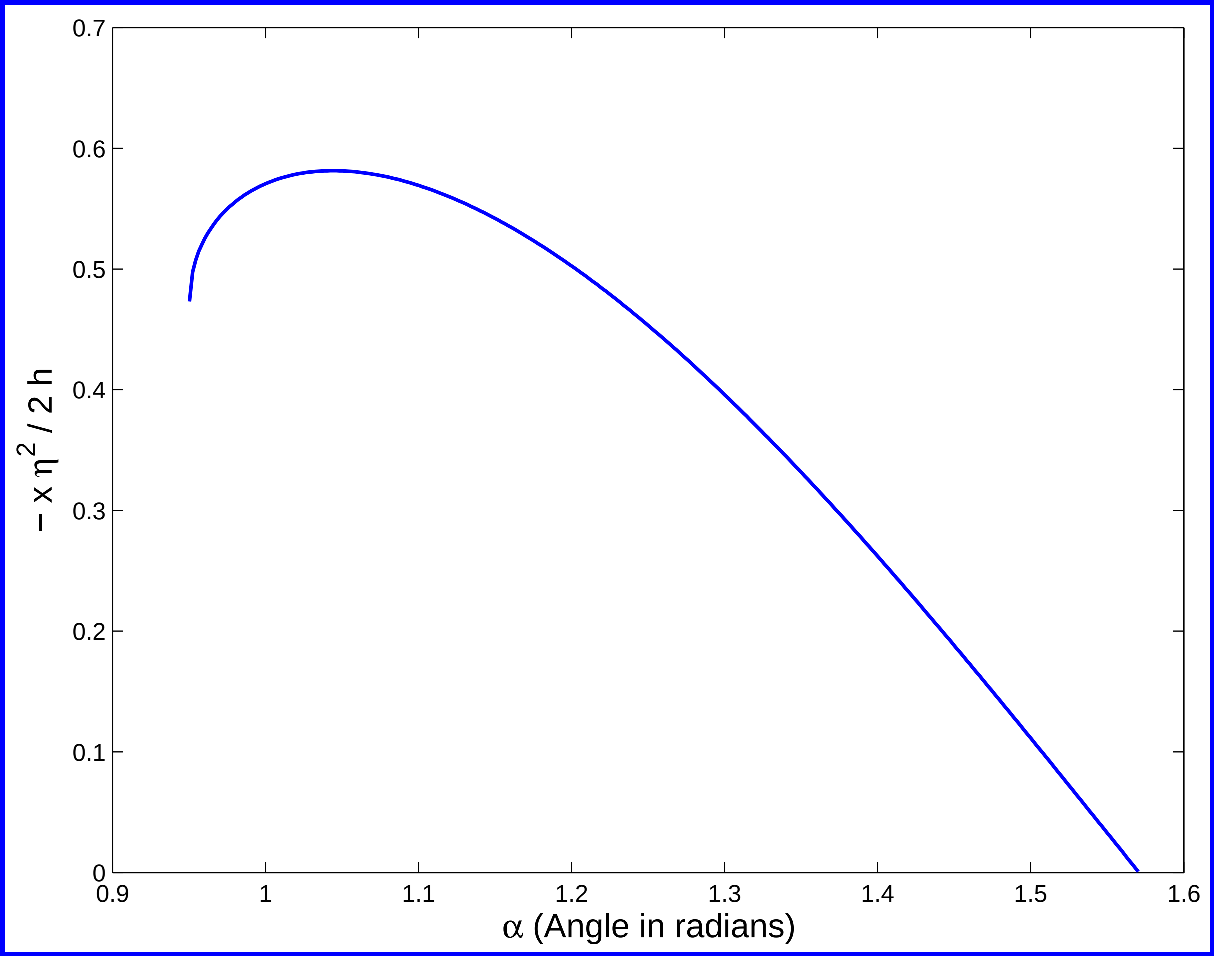
The plus sign yields a larger value for x and thus must correspond to the case where the projectile hits the target on the way down, and the minus sign yields a smaller value for x and therefore corresponds to the case where the projectile hits the target on the way up. To find the maximum distance we will therefore choose the plus sign.

If we measure lengths in units of h, we see that the problem has a single dimensionless parameter, which we choose to be $\eta^2 = \frac{2gh}{v^2}$. In terms of a dimensionless horizontal distance $\chi = x\eta^2/(2h)$, we see that

$$\chi = \sin(\alpha)\cos(\alpha) + \cos(\alpha)\sqrt{\sin^2(\alpha) - \eta^2}.$$

Notice that if η is larger than 1 then there cannot be any solution since the quantity inside the square root is negative. [When $\eta > 1$, we have $v^2 < 2gh$. Multiplying the inequality by $2m$, where m is the mass of the projectile, gives $\frac{1}{2}mv^2 < mgh$. This means that the initial kinetic energy of the projectile is less that its gravitational potential energy at the same height of the target. Therefore the projectile can never hit the target.] We will therefore assume from here on that η is less than or equal to 1.

Even then χ is real only if $\alpha > \alpha_0 = \sin^{-1}(\eta)$. It is clear that we do not want to shoot the projectile backward and therefore α must be less than $\pi/2$. Therefore we want to consider α between α_0 and $\pi/2$. Using Matlab we can easily plot χ for α between α_0 to $\pi/2$. We see that there is indeed a maximum for α around 1.04.



To locate this maximum more accurately, we can use any one of our one-dimensional minimization programs. However we want to keep $\alpha \in [\alpha_0 \pi/2]$, and so we may want to use the golden section method with α_0 and $\pi/2$ as the lower and upper bounds. Maximizing χ with respect to α is the same as minimized $-\chi$ with respect to α.

We use the following parameter values as given in the problem: $h = 13.5m$, $g = 9.8065m/s^2$, and $v = 20m/s$. The result computed to the maximum accuracy is $\alpha = 1.04414289709282$. The error is expected to be proportional to the square root of the machine epsilon.

This means that it should be accurate to about 7 significant figures. For this value of α we obtain $\chi = 0.58143034836513$. Converting back to the actual distance, we find that the maximum distance is given by $x = 2h\chi/\eta^2 = 23.7161208735077$1.

Exact Analytical Solution

It turns out that the problem can be completely solved analytically using nothing more than simple differentiation and manipulating trigonometric identities. You are definitely not required to perform the following analysis but I hope you are a little curious to see how that can be done.

Since χ depends on only a single variable α, we can differentiate χ with respect to α and set the result to zero to try to find the maximum. It is important to simplify the expressions as much as possible to reduce the amount of work that we need to do. Otherwise we may not be able to solve the resulting transcendental equation.

First we use the following trigonometric identities

$$\sin(\alpha)\cos(\alpha) = \frac{1}{2}\sin(2\alpha),$$

$$\cos^2(\alpha) = \frac{1}{2}(1 + \cos(2\alpha))$$

so that

$$\chi = \frac{1}{2}\left(\sin(2\alpha) + \sqrt{\sin^2(2\alpha) - 2\eta^2(1 + \cos(2\alpha))}\right).$$

Differentiate it with respect to 2α and setting the result to 0 yields

$$\cos(2\alpha) + \frac{\sin(2\alpha)\cos(2\alpha) + \eta^2 \sin(2\alpha)}{\sqrt{\sin^2(2\alpha) - 2\eta^2(1 + \cos(2\alpha))}} = 0.$$

Moving one of the terms to the other side of the equation, squaring both sides and simplifying the expression by replacing $\sin^2(2\alpha)$ by $1 - \cos^2(2\alpha)$ to obtain

$$(\eta^2 - 2)\cos^2(2\alpha) - 2\cos(2\alpha) - \eta^2 = 0.$$

Notice that $\cos(2\alpha) = -1$ is a solution of this equation.

This solution is clearly not acceptable.

The other factor can be obtained by synthetic division. Thus we have

$$(\cos(2\alpha) + 1)((\eta^2 - 2)\cos(2\alpha) - \eta^2) = 0;$$

Therefore the other root is

$$\cos(2\alpha) = \frac{\eta^2}{\eta^2 - 2}.$$

This gives the solution

$$\alpha = \frac{1}{2}\cos^{-1}\left(\frac{\eta^2}{\eta^2 - 2}\right).$$

From this result we find that

$$\sin(2\alpha) = \sqrt{1 - \cos^2(2\alpha)} = 2\frac{\sqrt{1-\eta^2}}{2-\eta^2}.$$

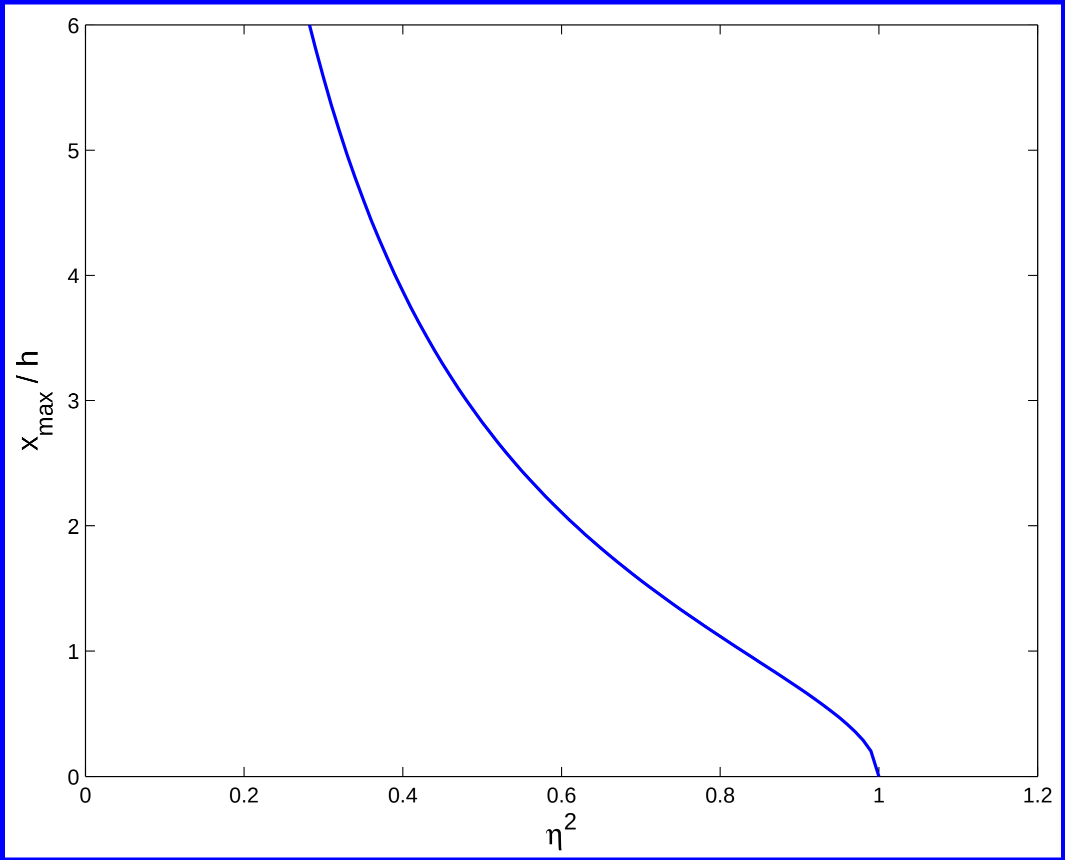
Substituting the expressions for $\cos(2\alpha)$ and $\sin(2\alpha)$ into the expression for χ, and simplifying the resulting expression yields the very simple result

$$\chi = \sqrt{1 - \eta^2},$$

at the maximum point. Therefore the maximum distance is given by

$$x = 2h\frac{\sqrt{1 - \eta^2}}{\eta^2}.$$

This result is shown in the following plot.



[We are not going to show that the extremum point is in fact a maximum.] Numerical results that we obtained earlier agree well with the above exact analytical results.