SOLUTION FOR ASSIGNMENT 1

Problem 1

(a) The characteristic polynomial is given by

$$\det \begin{pmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = 0.$$

(b) The roots of the characteristic equation are given by

$$1 - \lambda = \pm 2,$$

therefore they are given by 3 and -1.

(c) Thus the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$.

(d) For eigenvalue $\lambda_1 = 3$, we obtain from the first equation of $\mathbf{Ax} = \lambda\mathbf{x}$ the relation

$$(1-3)x_1 + 4x_2 = 0,$$

from which we have $x_1 = 2x_2$. The eigenvector is then given by

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}.$$

if we use the infinite norm, and by

$$\mathbf{x}_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}.$$

if we use the Euclidean norm.

For eigenvalue $\lambda_2 = -1$, we obtain from the first equation of $\mathbf{Ax} = \lambda\mathbf{x}$ the relation

$$(1+1)x_1 + 4x_2 = 0,$$

from which we have $x_1 = -2x_2$. The eigenvector is then given by

$$\mathbf{x}_2 = \begin{bmatrix} 1 \\ \frac{-1}{2} \end{bmatrix}.$$

if we use the infinite norm, and by

$$\mathbf{x}_2 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{-1}{\sqrt{5}} \end{bmatrix}.$$

if we use the Euclidean norm.

(e) We use the normalized power iteration with starting vector

$$\mathbf{x}^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The first iteration gives

$$\mathbf{A}\mathbf{x}^{(0)} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

This gives the vector

$$\mathbf{x}^{(1)} = \begin{bmatrix} 1 \\ \frac{2}{5} \end{bmatrix}$$

using the infinite norm.

The second iteration gives

$$\mathbf{A}\mathbf{x}^{(1)} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} \frac{13}{5} \\ \frac{7}{5} \end{bmatrix}.$$

This gives the vector

$$\mathbf{x}^{(2)} = \begin{bmatrix} 1 \\ \frac{7}{13} \end{bmatrix}$$

using the infinite norm. The vector is clearly approaching the dominant eigenvector $\mathbf{x}_1$.

(f) Result from a numerical calculation using a tolerance of 1×10^{-4} is shown below.

it	eigenvalue	eigenvector	
1	3.6172	1.3823	0.55291
2	2.8294	1.2702	0.68397
3	3.0603	1.309	0.63856
4	2.9803	1.2963	0.65349
5	3.0066	1.3006	0.64849

```
           6        2.9978        1.2991       0.65016
           7        3.0007        1.2996        0.6496
           8        2.9998        1.2994       0.64979
           9        3.0001        1.2995       0.64973
          10             3        1.2995       0.64975
evec =
      1.2995
     0.64975
eval =
           3
```

This eigenvector is not properly normalized. If we normalize it according to the infinite norm:

```
evec/(max(abs(evec)))
ans =
           1
     0.50001
```

we indeed obtain convergence to the dominant eigenvector $\mathbf{x}_1$.

(g) Using the approximate eigenvector

$$\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

the Rayleigh quotient is

$$\lambda = \frac{\mathbf{x}^T\mathbf{A}\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \frac{\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \frac{\begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix}}{2} = \frac{7}{2} = 3.5.$$

The resulting approximation for the eigenvalue is getting close to 3.

(h) Inverse iteration using the function InverseIterationF.m yields the result:

```
     it             eigenvector             eigenvalue
           1        1.0405             0       0.96104
           2        1.0845       -1.0845      -0.31982
           3        1.0577      -0.42307       -1.7089
```

```
           4       1.0638    -0.57279    -0.86172
           5       1.0615    -0.51782     -1.0535
           6       1.0622    -0.53551    -0.98308
           7        1.062    -0.52955     -1.0057
           8       1.0621    -0.53153     -0.9981
           9       1.0621    -0.53087     -1.0006
          10       1.0621    -0.53109    -0.99979
          11       1.0621    -0.53101     -1.0001
          12       1.0621    -0.53104    -0.99998
evec =
      1.0621
    -0.53104
eval =
          -1
```

Normalizing the eigenvector using the infinite norm gives

```
evec/(max(abs(evec)))
ans =
           1
    -0.50001
```

which is very close to the other eigenvector $\mathbf{x}_2$.

(i) Inverse iteration with a shift $\sigma = 2$ using the function ShiftedInverseIterationF.m yields

```
       it       eigenvector            eigenvalue
           1       1.5097     0.60389       1.104
           2       1.3508     0.72737     0.96861
           3       1.4049     0.68533      1.0108
           4        1.387     0.69924     0.99644
           5        1.393     0.69459      1.0012
           6        1.391     0.69614      0.9996
           7       1.3917     0.69562      1.0001
           8       1.3914      0.6958     0.99996
           9       1.3915     0.69574           1
evector =
     0.89443
      0.4472
```

```
evalue =
              3
```

We have convergence to the eigenvalue $\lambda_1 = 3$. If the eigenvector is normalized according to the infinite norm, then we have

```
 evector/(max(abs(evector)))
ans =
              1
        0.49998
```

which is very close to the eigenvector $\mathbf{x}_1$. This is expected since the distance from $\lambda_1(= 3)$ to $\sigma(= 2)$ is 1, while the distance from $\lambda_2(= -1)$ to $\sigma(= 2)$ is 3. The iteration indeed converges to the eigenvector whose eigenvalue is closest to σ.